Electromagnetic Field Analysis in Rotational Electric Machines Using Finite Element — Analytical Hybrid Method

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Abstract—**A method which combines finite element with analytical solutions for solving electromagnetic field problems in rotational electric machines is explored. The principle of this method and procedure of applying this hybrid method are presented with explicit equations. Two examples are given to show the effectiveness of the method proposed.**

I. Introduction

Electromagnetic (EM) field problems in electric machines can be solved by using analytical or numerical methods such as the finite element (FE) method. The major advantages of the analytical solution are its accuracy and convenience in field analysis. But these advantages may only be exploited fully when the geometry of the machine is simple and the materials involved have linear characteristics. The process of obtaining an analytical expression is usually difficult, even for much simplified mathematical models derived from real machines. On the other hand, the FE method is a powerful tool for analyzing EM field problems even if the materials are non-linear and the geometries are complex. However, the accuracy of the analysis with FE method is sensitive to errors induced by the FE modelling. This causes problems in some post-processing analyses such as the EM torque evaluation for electric machines.

The field analysis for electric machines is a complicated process due to the non-linear characteristics of the materials and the complex machine structure. However, as the field in some regions of the machine can be described by using analytical expressions, such as the airgap of the machine, it is possible to combine the FE method with an analytical method to solve the overall field problem in the machine. In this way, both the advantages of the analytical method and FE method can be utilized in the solution. This paper will discuss this idea to develop "Finite element-Analytical Hybrid method".

II. The Vector Magnetic Potential in Airgap

In a 2-D magnetic field problem, the vector magnetic potential **A** contains only the z direction component A_z, that is, $\mathbf{A}=A_z\cdot\bar{u}_z$. For simplicity, the subscript, z, will be omitted in the following analysis.

If there is no current source in the airgap of the electric machine, the magnetic field can be described with a Laplace equation. When the polar coordinates are used, we have

$$\frac{\partial^2 A}{\partial r^2}+\frac{1}{r}\frac{\partial A}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2}=0 \tag{1}$$

for the airgap, $R_1<r<R_2$, where R_1 and R_2 are the inner and outer radius of the airgap, so the general solution is

$$A(r,\theta)=\sum_{n=1}^{\infty}[(A_n r^n+B_n r^{-n})\cos(n\theta)+(C_n r^n+D_n r^{-n})\sin(n\theta)]+A_0\ln(r)+B_0 . \tag{2}$$

where, A_n, B_n, C_n, and D_n, are constants, and θ is expressed in electrical degrees. It will be noted that A_n, B_n, C_n, and D_n are four independent coefficients for a given "n" in the equation.

When the FE solutions in other regions have been obtained, the potential $A|_{r=R1}$ and $A|_{r=R2}$ are known, they can be described by Fourier expressions:

$$A(R_1,\theta)=a_0+\sum_{n=1}^{\infty}[a_n\cos(n\theta)+b_n\sin(n\theta)] \tag{3.1}$$

$$A(R_2,\theta)=c_0+\sum_{n=1}^{\infty}[c_n\cos(n\theta)+d_n\sin(n\theta)] \tag{3.2}$$

Comparing the coefficients in (2) and (3), it yields

$$A_n=(a_nR_2^{-n}-c_nR_1^{-n})/\Delta_n \tag{4.1}$$

$$B_n=(c_nR_1^{n}-a_nR_2^{n})/\Delta_n \tag{4.2}$$

$$C_n=(b_nR_2^{-n}-d_nR_1^{-n})/\Delta_n \tag{5.1}$$

$$D_n=(d_nR_1^{n}-b_nR_2^{n})/\Delta_n \tag{5.2}$$

$$A_0=(a_0-c_0)/\ln(R_1/R_2) \tag{6.1}$$

$$B_0=[c_0\ln(R_1)-a_0\ln(R_2)]/\ln(R_1/R_2) \tag{6.2}$$

In (4) and (5), $\Delta_n=R_1^n\cdot R_2^{-n}-R_2^n\cdot R_1^{-n}$, and $n\neq0$.

III. Field Analysis Using A Hybrid Method

For convenience in discussion, the Neuman condition is considered as zero, i.e., $\partial A/\partial n=0$. Therefore, the variational formulation of the field problem can be expressed by minimizing the energy function:

$$W=\frac{1}{2}\iint_{\Omega_a}\nu_0B^2d\Omega+\iint_{\Omega_s}[\int_0^B H(B)dB-J\cdot A]d\Omega+\iint_{\Omega_r}[\int_0^B H(B)dB-J\cdot A]d\Omega=W_a+W_s+W_r \tag{7}$$

where, Ω_a, Ω_s and Ω_r denote the regions of airgap, stator and rotor, H and B are the magnetic intensity and flux density.

The last two terms on the right-side of (7) can be described

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conveniently by using FE expressions. Using the analytical expression (2), the first term, W_a, can also be described by an analytical expression as follows

$$W_a = \frac{\pi\nu_0(a_0^2+c_0^2-2a_0c_0)}{\ln(R_2/R_1)} + \frac{\pi\nu_0}{2}\sum_n \frac{n(a_nc_n+b_nd_n)}{R_1^{2n}-R_2^{2n}} \cdot[(a_n^2+b_n^2+c_n^2+d_n^2)(R_1^{2n}+R_2^{2n})-4R_1^nR_2^n(a_nc_n+b_nd_n)]. \quad (8)$$

In FE analysis, the potential A is expressed by a set of discrete values $[A_i]$ which are the potentials on the FE nodes in the problem domain, and the variational formulation can be described by the following equation

$$[\frac{\partial W}{\partial A_i}]=[\frac{\partial W_s}{\partial A_i}]+[\frac{\partial W_r}{\partial A_i}]+[\frac{\partial W_a}{\partial A_i}]=[0] \quad (9)$$

where, $i=1,2,\cdots,N$, and N is the total number of the nodes.

Equation (9) leads to an algebraic equation:

$$[S_{ij}]\cdot[A_j]=[f_i] \quad (10)$$

In the above equation, the order of the stiffness matrix $[S_{ij}]$ is $N\times N$, the orders of $[A_j]$ and $[f_i]$ are the same, i.e., $N\times 1$.

The expressions for W_s' and W_r' can be deduced by using FE theory directly, they contribute to $[S_{ij}]$ and $[f_i]$ of (10). If the materials in the machine are non-linear, these contributions make the equation non-linear [1,2,3]. It should be noted, if W_a' does not contribute to (10), $[S_{ij}]$ is actually formed by two independent sub-matrixes.

When the FE method is used, the potential A on the airgap boundary is described by a set of discrete values $[A_i]^{(b)}$. It is clear, $[A_i]^{(b)}\subset[A_i]$. When the whole airgap region is used in the analysis and linear interpolation function is used to construct a continual function from the $[A_i]^{(b)}$, it yields

$$a_n=\frac{-1}{n^2\pi}\sum_{i=1}^{M_1}A_i[\frac{\sin(n\theta_{i12})\sin(nh_{i12})}{h_{i12}}-\frac{\sin(n\theta_{i23})\sin(nh_{i23})}{h_{i23}}] \quad (11)$$

$$b_n=\frac{1}{n^2\pi}\sum_{i=1}^{M_1}A_i[\frac{\cos(n\theta_{i12})\sin(nh_{i12})}{h_{i12}}-\frac{\cos(n\theta_{i23})\sin(nh_{i23})}{h_{i23}}] \quad (12)$$

and

$$a_0 = \frac{1}{4\pi}\sum_{i=1}^{M_1}(h_{i12}+h_{i23})\cdot A_i \quad (13)$$

where, $\theta_{i12}=\frac{1}{2}(\theta_i+\theta_{i-1})$, $\theta_{i23}=\frac{1}{2}(\theta_i+\theta_{i+1})$, $h_{i12}=\frac{1}{2}(\theta_i-\theta_{i-1})$, and $h_{i23}=\frac{1}{2}(\theta_{i+1}-\theta_i)$. M_1 is the number of the nodes on the boundary $r=R_1$.

In the same way, the expressions for c_n and d_n can also be obtained. In these equations, the coefficients a_n, b_n, c_n and d_n are functions of $[A_i]^{(b)}$, therefore, W_a shown in (8) is also a function of $[A_i]^{(b)}$.

The expression of $\partial W_a/\partial A_i$ $(i=1,2,\cdots,N)$ can be deduced from (8) directly, and they also contribute to the stiffness matrix $[S_{ij}]$. Thus, the effects of each potential A_i are related by (10). In this way, equation (10) is formed by FE method and analytical method, and it yields the proposed "Finite element-Analytical Hybrid Method" (**FAHM**).

In the application of FAHM, the fields in the rotor and stator of the machine are described by two independent FE expressions. The EM relation between the rotor and stator is implicated in the analytical expression for the airgap field. It is clear, this feature will simplify the process of setting up reasonable FE meshes. By using FAHM, it is also convenient to realize the dynamic analyses for rotational machines, as the intricate procedure for mesh adjustment in the airgap is avoided and the discretization error induced by the mesh variation is eliminated. The analytical expression for the airgap is also very helpful for some post-processing analysis such as for the EM torque evaluation. The applicability of FAHM can be further extended to cases where an analytical expression can be found for the field in the rotor or the stator whilst the rest of the machine has to be described by a FE expression. In comparison with the direct FE method (**DFEM**), this strategy can improve the accuracy of the analysis and reduce the computing time further as the process for subdividing the geometry is simplified and the number of the nodes is reduced. In the following section, examples will be used to demonstrate these features of FAHM.

IV. Applications

For better understanding of the principle and effectiveness of FAHM, two computational examples are given in this section.

A. Example 1

The problem domain is an annular segment, $3\le r\le 8$. The boundary condition is as follows:

$$\begin{cases} A(r,\theta)|_{r=3} = 0 & (14.1) \\ A(r,\theta)|_{r=8} = \cos(2\theta) & (14.2) \end{cases}$$

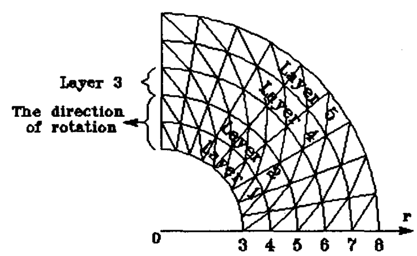

Fig.1 The problem domain and the mesh used for FE analysis

Fig.1 shows the domain used for the analysis. Thus, the boundary conditions at $\theta=0$ and $\theta=90°$ are half-periodic, i.e., $A(r,\theta)|_{\theta=0°}=-A(r,\theta)|_{\theta=90°}$. The domain is subdivided by the mesh shown in this figure. When Layer 3 is considered as the "airgap" and its energy function is expressed by (8), the solution at $r=6$ is shown in Table I. For comparison, the results obtained from DFEM based on the mesh of Fig.1 are also shown in the table.

In Table I, AN, DFE and FAH represent the analytical solution, direct FE solution and FAH solution, respectively.

To investigate the influence of the distortion of the "airgap" elements, the elements in Layer 1 and Layer 2 are rotated. The subscripts of DFE and FAH in Table I show the degree of rotation. Based on the boundary conditions (14), it can be seen that the field solution at r=6 will not be affected by the rotation.

TABLE I THE ANALYSIS RESULTS FOR EXAMPLE 1

	0°	10°	20°	30°	40°
AN	0.5380	0.5055	0.4121	0.2690	0.09342
FAH_0	0.5380	0.5055	0.4121	0.2690	0.09342
DFE_0	0.5373	0.5049	0.4116	0.2687	0.09331
FAH_5	0.5380	0.5055	0.4121	0.2690	0.09342
DFE_5	0.5070	0.4764	0.3884	0.2535	0.08804

Table I shows that the accuracy of FAHM is better than that of DFEM. The distortion of the "airgap" elements caused by the rotation can affect the accuracy of the solution obtained from DFEM, whilst the solution obtained from FAHM does not rely on the mesh in the "airgap" and its accuracy is not affected. The stability of the solution is important for some post-processing analysis, such as the dynamic EM torque evaluation [4]. From this point of view, FAHM is an effective method in EM field analysis for rotational machines.

B. Example 2

A slotless spindle motor is shown in Fig.2. The boundary conditions at $\theta=0$ and $\theta=180°$ are full-periodic, i.e., $A(r,\theta)|_{\theta=0°}=A(r,\theta)|_{\theta=180°}$.

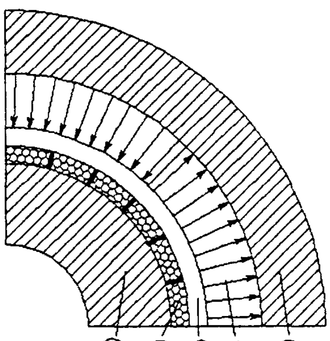

Fig.2 The range of the spindle motor used in the field analysis
1. stator yoke; 2. armature coils; 3. airgap; 4. PM ring; 5. rotor yoke.

In this example, the stator core is assumed to be linear and rotor core is non-linear, so W_s in (3) can also be expressed as an analytical expression and W_r is discribed by using a FE mesh. As modification of the airgap elements is required during computation, it is not convenient to obtain the Torque-Position Characteristic curve for a fixed current distribution by using DFEM. When FAHM is used, the computation can be done easily as this modification is not required, and the result is shown in Fig.3.

For some rotor positions, the results obtained from DFEM with high density mesh can be considered as the reference to check the accuracy of other methods. A comparison is made for the torque at $\theta=90°$, and the results are as follows

DFEM: $T(90°)=10.047$ mNm
FAHM: $T(90°)=10.064$ mNm

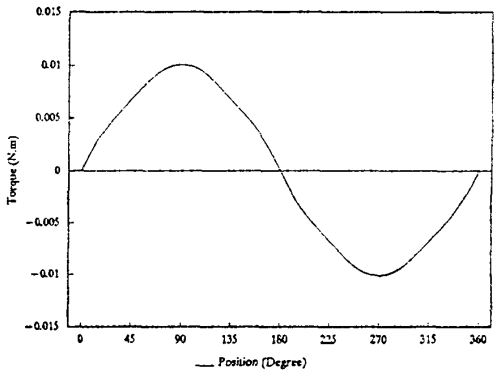

Fig.3 The Torque-Position Characteristic of the spindle motor

V. CONCLUSIONS

In the application of the FE-Analytical Hybrid method proposed in the paper, the rotor and stator regions are subdivided independently, and an analytical expression is used to describe the field in the airgap. As the fields of the rotor and stator are coupled by the analytical expression of the airgap field, the formulation of the total field problem can also be established. Thus, the pre-processing in the FE analysis can be simplified. The concept of FAHM can be extended to cases where the field in the rotor or stator can be described by an analytical expression but the other part of the machine has to be described by a finite element expression. The examples used in the paper show that the using of the FAHM improves the accuracy of the field analysis as the analytical solution plays an important role in the computation. Furthermore, intensive calculations such as the dynamic torque analysis for rotational electric machines can be realized conveniently.

VI. REFERENCES

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